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LETTER TO THE EDITOR

Lame equation, sl(2) algebra and isospectral deformations

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Abstract. It is proved that the n-zone Lame equation is equivalent to a spectral problem for a quadratic element of a universal enveloping sl(2) algebra. The Riemannian surface corresponding to the eigenstates in a parametric space forms the $(2n+1)$-sheet surface which splits in four subsurfaces; three of them contain the same number of sheets.

Let us take the n-zone Lame equation

$$-\mathrm{d}^2\psi/\mathrm{d}x^2+n(n+1)\mathscr{P}(x)\psi=\lambda\psi \tag{1}$$

where $\mathscr{P}(x)$ is the Weierstrass function in a standard notation, namely a double-periodic meromorphic function for which the equation $\mathscr{P}'^2=(\mathscr{P}-\theta_1)(\mathscr{P}-\theta_2)(\mathscr{P}-\theta_3)$ holds. It is known [1] that instead of equation (1) the new equation emerges

$$\eta''+\frac{1}{2}\left(\frac{1}{\xi-a_1}+\frac{1}{\xi-a_2}+\frac{1}{\xi-a_3}\right)\eta'-\frac{n(n+1)\xi+\lambda}{4(\xi-a_1)(\xi-a_2)(\xi-a_3)}\eta=0 \tag{2}$$

if the new variable $\xi=\mathscr{P}(x)+\frac{1}{3}\sum a_i$ is introduced, where $\eta(\xi)\equiv\psi(x)$. Here the new parameters a_i satisfy the system of linear equations $e_i=a_i-\frac{1}{3}\sum a_i$. Equation (2) is named by an algebraic form for the Lame equation. There exists a spectral parameter λ for which equation (2) has four types of solutions:

$$\eta^{(1)}=P(\xi) \tag{3}$$

$$\eta_i^{(2)}=(\xi-a_i)^{1/2}P(\xi) \qquad i=1,2,3 \tag{4}$$

$$\eta_i^{(3)}=(\xi-a_l)^{1/2}(\xi-a_m)^{1/2}P(\xi) \qquad l\neq m;\ i\neq l,m;\ i=1,2,3 \tag{5}$$

$$\eta^{(4)}=(\xi-a_1)^{1/2}(\xi-a_2)^{1/2}(\xi-a_3)^{1/2}P(\xi) \tag{6}$$

where $P(\xi)$ is a polynomial in ξ. If the value of parameter n is fixed, there are $(2n+1)$ linear independent solutions of the following form: if $n=2k$ is even, then the $\eta^{(1)}(\xi)$ and $\eta^{(3)}(\xi)$ solutions arise, if $n=2k+1$ is odd we have solutions of the $\eta^{(2)}(\xi)$ and $\eta^{(4)}(\xi)$ types.

Theorem 1. The spectral problem (1) with polynomial solutions (3)-(6) is equivalent to the spectral one for the quadratic element of universal enveloping sl(2) algebra:

$$\{-a_{+0}J^+J^0-a_{+-}J^+J^--a_{00}J^0J^0-a_{0-}J^0J^--a_{--}J^-J^- +b_+J^++b_0J^0+b_-J^-\}\eta^{(i)}=\lambda\eta^{(i)} \tag{7}$$

where

$$a_{+0}=4 \qquad a_{+-}+a_{00}=-4\sum a_i \qquad a_{0-}=4\sum a_ia_j \qquad a_{--}=a_1a_2a_3. \tag{8}$$

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The parameters b_i depend upon the type of solutions (3)-(6) and are equal to

$$b_+ = -6k-2 \qquad b_0 = 4(k+1)\sum a_i + a_{00} \qquad b_- = -2(k+1)\sum a_i a_j \tag{3a}$$

$$b_+ = -6k-6 \qquad b_0 = 4(k+2)\sum a_i + a_{00} - a_i$$

$$b_- = -2(k+1)\sum a_i a_j - 4a_l a_m \qquad i \neq l, m \tag{4a}$$

$$b_+ = -6k-4 \qquad b_0 = 4(k+1)\sum a_i + a_{00} + 4a_i$$

$$b_- = -2(k+2)\sum a_i a_j + 4a_l a_m \qquad i \neq l, m \tag{5a}$$

$$b_+ = -6k-8 \qquad b_0 = 4(k+2)\sum a_i + a_{00} \qquad b_- = -2(k+2)\sum a_i a_j. \tag{6a}$$

Here $J^{\pm,0}$ are the sl(2) generators

$$J^+ = \xi^2 \,\mathrm{d}/\mathrm{d}\xi - 2j\xi \qquad J^0 = \xi\,\mathrm{d}/\mathrm{d}\xi - j \qquad J^- = \mathrm{d}/\mathrm{d}\xi$$

where

$$\begin{aligned} &j = \tfrac{1}{2}k && \text{for (3) and (4)} \\ &j = \tfrac{1}{2}k - \tfrac{1}{2} && \text{for (5) and (6)} \end{aligned} \tag{9}$$

is a spin of the representation considered.

The validity of this theorem can be checked by substituting the expressions for generators into equation (7) provided conditions (3a)-(6a), (8) and (9) are satisfied.

So, each type of solution (3)-(6) corresponds to the particular spectral problem (7) with a special set of parameters. It can be easily shown that the calculation of eigenvalues λ corresponds to the solution of a characteristic equation for the four-diagonal matrix:

$$\begin{aligned} A_{l,l-1} &= (l-1-2j)[4(j+1-l)+b_+] \\ A_{l,l} &= [l(2j+1-l)(a_{+-}+a_{00}) - (l+j^2)a_{00} + (l-j)b_0] \\ A_{l,l+1} &= (l+1)(j-l)a_{0-} + (l+1)b_- \\ A_{l,l+2} &= -(l+1)(l+2)a_{--}. \end{aligned} \tag{10}$$

The size of this matrix is $(k+1)\times(k+1)$ for (3) and (4) and $k \times k$ for (5) and (6). As a consequence of theorem 1 it is possible to prove a second theorem.

Theorem 2. The parametric $(2n+1)$-sheet Riemannian surface of eigenvalues of equation (1) in parameter a_i at fixed parameters n and a_j $(j \neq i)$ contains four disconnected pieces: one of them corresponds to $\eta^{(1)}(\eta^{(4)})$ solutions and the others correspond to $\eta^{(3)}(\eta^{(2)})$. At $n=2k$ the Riemannian subsurface for $\eta^{(1)}$ has $(k+1)$ sheets and the number in each of the others is k. At $n=2k+1$ the number of sheets for $\eta^{(4)}$ is k and for $\eta^{(2)}$ each surface contains $(k+1)$ sheets.

It is worth emphasising that we cannot find a relation between the two-zone potential

$$V = -2\sum_{i=1}^{3} \mathscr{P}(x-x_i) \qquad \sum_{i=1}^{3} x_i = 0 \tag{11}$$

(see [2]) and the spectral problem (7) at $n=2$ (potential (11) is related to the Lame potential (1) by isospectral deformation). In this case eigenvalues λ do not depend on parameters a_{00}, a_{--} (see equation (8)).

However, at $n=2$ there is a spectral deformation (1) other than (11). It arises from the fact that the addition to the operator (7) of the term $a_{++}J^{-}J^{+}$ does not change the characteristic matrix (10). To obtain its explicit form, one can substitute the generators $J^{\pm,0}$ into equation (7) with the additional term above. The resulting equation then reduces to the Schrödinger form (compare [3]). As a final result, we obtain

$$V(x)=a_{++}[2a_{++}\xi^6-(a_{+-}+a_{00})\xi^4-2a_{0-}\xi^3-3a_{--}\xi^2]/P_4^2(\xi)+P_2(\xi) \tag{12}$$

where

$$x=\int\frac{d\xi}{\sqrt{P_4(\xi)}} \qquad \begin{aligned} P_4(\xi)&=a_{++}\xi^4+a_{+0}\xi^3+(a_{+-}+a_{00})\xi^2+a_{0-}\xi+a_{--}\\ P_2(\xi)&=-n(n+1)\xi+\frac{a_{00}}{4}+\frac{b_0}{2}.\end{aligned} \tag{13}$$

In general, the potential (12) contains four double poles in x and does not reduce to (11). It is worth noting that the eigenfunctions for (12) have the form

$$\psi(x)=\left\{\frac{A\xi+B}{(\xi-a_i)^{1/2}(\xi-a_j)^{1/2}}\right\}\exp\left(-a_{++}\int\frac{\xi^3\,d\xi}{P_4(\xi)}\right). \tag{14}$$

Here ξ is given by (13). The first five eigenvalues of the potential (12) do not depend on the parameters a_{00}, a_{--}, a_{++}.

In conclusion I would like to acknowledge I M Gel'fand and S P Novikov for suggesting the problem and I M Krichiver and B A Dubrovin for useful discussions.

Note added in proof. After submitting this work for publication, I learned about the paper [4] and references therein with a rather similar representation (see (7)) of the Lame equation but in a Jacobi form. In addition, quasi-exactly-solvable problems associated with the Lame equation were also discovered in [4]. All of them belong to a general class of one-dimensional quasi-exactly-solvable problems described in [3].

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